MODELS FOR PACKING CIRCULAR OBJECTS INTO RECTANGULAR SPACES

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ABSTRACT

Novel mathematical models for packing identical circular objects into rectangular spaces are here presented. The study explores on different packing patterns that tend to increase the population density of a given rectangular space by way of systematic repositioning of the objects and by applications of some trigonometric concepts in determining the effect of repositioning to the vertical distances between the centers of the objects across the contiguous rows. The results showed that if the dimension (rc) of a rectangular space is rc = 8x5, where the unit of measure of the space is the diameter of a circular object, then the default arrangement of the objects can be repositioned so that the content of the space is maximum. The results also showed that a rectangular space attains its maximum content if row, r, is a multiple of $\lceil\frac{7.464}{2}\rceil$ and column c ≥ 5. In order to determine whether the population density of a rectangular space can be increased by applying some packing patterns, two mathematical models are developed, through which the exact number of objects that can be accommodated in a space is calculated. This study shows that there are deterministic mathematical models of calculating the maximal number of identical circular objects that can be packed into rectangular spaces. In cases, however, where the rectangular container provides empty space either on the row or column or both with length less than the diameter of one circular object, then adjustment on the models may be made. Hence, it is recommended that such particular cases have to be further explored in future study.

Keywords: packing models, circular objects and rectangular spaces

INTRODUCTION

As the 21st century unfolds, we see the rapid evolution of human civilization whose landscape is primarily defined by revolutionary breakthroughs in science and technological development vis-à-vis population explosion. As human population increases day by day resulting to rapid increase in the demand for the use of machines, appliances and gadgets of all kinds that tend to occupy significant livable space at home and to the rapid depletion of available arable space, technological advances have been on the development of machines and gadgets that address space saving concerns and multiple capability needs. Consequently, many machines, appliances and gadgets are now built with minimum sizes and with multiple task capabilities. Television sets, computers, communications and musical gadgets are few examples of technological products that have been miniaturized to save on space but with built-in capabilities far exceeding their much bigger sized predecessors.

The idea of saving space in technological development has been extended to many human activities such as maximizing land use, increasing population densities of objects, maximizing content of boxes, etc.

This study attempted to address a specific concern on maximizing content of rectangular spaces when populated with circular objects.

It is said that one of the most scientifically challenging problems in operation research is packing circular and spherical objects in
some predetermined rectangular two and three dimensional spaces. This is partly because circular objects do not tessellate unlike other objects with straight edges. Packing circular and spherical objects are real world activities being undertaken in industries involved in the production, packing and transport of such products as textile, sports equipment, automobile parts, food and hardware products, to mention a few. Application of packing patterns may also be extended to agriculture. The population density of a rectangular planting area may be increased by applying certain packing pattern without compromising the required circular area allotted to each plant.

Packing circular objects is considered a “very interesting NP-hard combinatorial optimization problem” because there is “no procedure [that] is able to exactly solve them in deterministic polynomial time” (Hifi and M’Hallah, 2009).


The works of the above-mentioned researchers explored on possible ways of packing circular objects by determining the radii of circles that either optimize or maximize a given rectangular as well as circular container. Some of these works include determining the minimum perimeters of rectangles that enclose identical non-overlapping circles. The researchers developed several physical as well as electronic approaches of solving various packing circles problems ranging from “computer-aided optimality proofs, to branch-and-bound procedures, to constructive approaches, to multi-start nonconvex minimization, to billiard simulation, to multiphase heuristics and metaheuristics” (Hifi and M’Hallah, 2009). These approaches were applied to packing problems involving both identical and varied-sized circular objects.

Although the published paper presents thorough descriptions of the reviewed approaches, there was no illustration made showing how the approaches could be used to solve some particular packing problems.

This work presents a simple investigation on a special circle packing problem that deals with identical circular objects with fixed sizes and rectangular spaces whose dimensions’ unit of measure is the diameter of a circle. This special packing problem was not thoroughly considered in the above studies. In the foregoing studies, various sophisticated approaches were developed in solving circle packing problems by focusing on minimizing the length of the radii of both identical and non-identical circles in order to attain optimum covers of the interiors of given rectangular spaces. Although there are studies that deal with packing identical non-overlapping circles in squares, the aspect considered is the circles which were
randomly scattered into the square container. This study on the other hand, explored on non-random and smooth patterns of fitting the most number of identical fixed-sized circles in some pre-determined dimensions of rectangular spaces.

To ventilate the idea pursued in this study, consider the given rectangular packs above that are fully filled with circular objects in their default positions. The letters R and C denote the row and column of the rectangular space, respectively.

The dimension of the rectangular space in figure 1(a) is \( r \times c = 8 \times 4 \), for figure 1(b), \( 5 \times 9 \) and for figure 1(c), \( 8 \times 6 \).

If the objects are rearranged in ways different from their original positions such that no two contiguous rows are retained in their original positions, is it possible for these packs to exceed their original contents? The primary aim of this study is to develop some mathematical models that can be used to calculate the optimal number of equal-sized circular objects that can be fitted in a given rectangular spaces.

**METHODOLOGY**

The nature of the study is exploratory. The default positions of the circular objects in a two dimensional rectangular space (or \( r \times c \) space) require that the objects are arranged in rectangular position in which the objects are placed along the straight lines in both row \( r \) and column \( c \). The default position is altered, or repositioned, in some ways to form some new packs. The default pack is denoted by \( P_o \) while a new pack by \( P_n \).

The structural positions of the circles in \( P_n \) are mathematically analyzed in terms of determining the effect of the alteration on the vertical distance between the centers of the circles across any two contiguous rows in \( r \). The effect of the alteration on the content of \( P_n \), in consideration of the dimension of \( r \times c \) space, is also explored.

In order to determine whether \( P_n \) is tending to maximize the number of circles in \( r \times c \) space or tending to minimize it, the content of \( P_n \) is compared with that of \( P_o \). The minimum dimension of \( r \times c \) space that allows the addition of a layer in row \( r \) and that makes \( P_n \) exceeds the content of \( P_o \) is explored. Finally, the mathematical models of computing the contents of \( P_n \) are developed and proofs for their mathematical viability are provided.

**RESULTS AND DISCUSSION**

Given a fixed dimension of an \( r \times c \) space, the default pack (Fig 2a) when altered in certain way may yield the new pack shown in figure 2(b). The repositioning of the circles in figure 2(b) is a kind that tends to increase the number of layers in row \( r \) and the density at the middle part of the pack. This form of repositioning requires arranging the circles in triangular formation.

The effect of repositioning the circles in the manner as in figure 2(b) is row compression. In the default position shown in figure 3(a), the distance between the centers of the circles across any two

![Figure 1(a)](image1a.png) ![Figure 1(b)](image1b.png) ![Figure 1(c)](image1c.png)
contiguous rows is equal to the diameter of a circle. In the new position wherein each circle lies in between two circles below it, the vertical distance of the centers in the default position becomes a slant distance in the new position, as shown in figure 3(b) above. By applying tangent function in trigonometry it can be shown that the vertical distance \(d_o\) of the centers of the circles in their new position is \(\sqrt{3}/2 d\), which is 13.4\% shorter than the original vertical distance \(d_o\) of the centers across rows. This result means that, except in the first row, about 13.4\% of the original distance is lost in every row due to repositioning. Also, the result indicates that there exist a number \(n\) of rows in \(P_o\) in which the accumulated loss of distance will create enough space that will allow the addition of one more row in \(P_n\).

In order to determine this, the difference between the height \(H_o\) of \(P_o\) and the height \(H_n\) of \(P_n\), for some numbers \(n\) of rows, must be equal to the height \(d_n\) of the row to be inserted, which is \(\sqrt{3}/2 d\).

The height \(H_o\) of \(P_o\) is the product of the number of rows \(n\) and the diameter \(d\) of a circle. Thus,

\[
H_o = nd_o = nd \quad (1)
\]

Similarly, the height \(H_n\) of \(P_n\) is the total number \((n-1)\) of shortened vertical distance \(d_n\) times length of \(d_n\) which is \(\sqrt{3}/2 d\) plus the sum of the two un-shortened vertical distances which are the bottom radius \((1/2d)\) of the first row and the top radius \((1/2d)\) of the last row of circles in \(P_n\). Thus,

\[
H_n = (n-1)(\sqrt{3}/2d)+(1/2d+1/2d)=(n-1)(\sqrt{3}/2)d \quad (2)
\]

Finally, the number of rows in \(P_o\) that allows the addition of one more row in \(P_n\) is given in theorem 1.

**Theorem 1:** Let \(n \in \mathbb{Z}\) be the number of rows in \(P_o\) and \(T \in \mathbb{Z}\) be the minimum number of rows in \(P_o\) that accumulates a height enough to accommodate one more row in \(P_n\). Then

\[
T = \lceil n \rceil = \lceil 7.464 \rceil = 8
\]

**Proof:**

When the difference between \(H_o\) and \(H_n\) is equal to the height \((\sqrt{3}/2 d)\) of the row to be inserted, then a row can be added to \(P_n\). Thus,

\[
H_o - H_n = \sqrt{3}/2d
\]

Where: \(\sqrt{3}/2 d\) is, as shown above, the height \((d_n)\) of a row in \(P_n\).

\[
nd - (n-1)(\sqrt{3}/2d)+d=\sqrt{3}/2d \quad \text{From (1) and (2)}
\]

\[
n=2(2-\sqrt{3})= 7.464 \quad \text{(on solving)}
\]

So, the number of rows in \(P_o\) that accumulates a

![Figure 2(a)Figure 2(b)](Default Pack \(P_o\) New Pack \(P_n\))

![distance between centers](Figure 3(a) Figure 3(b))
height of $\sqrt{3}/2d$ in $P_n$ is 7.464. This is the minimum number ($T$) of rows that will allow the insertion of one more row in $P_n$. However, since $T$ is an integer, then there exist $n_1, n_2 \in \mathbb{Z}$ (where $n_2 = n_1 + 1$) such that if $n_1 < n < n_2$, then either $T = n_1$ or $T = n_2$. Because $n_1 < n$ implies that $T \neq n_1$. So, $T = n_2$. $T$ is the least integer $\geq n$. Thus, $T = \lceil n \rceil = \lceil 7.464 \rceil = 8$

From the result above it can be concluded that the number of rows that can be repositioned to create enough space for the addition of a desired number, $x$, of rows is the least integer greater than the multiple of 7.464. This assertion is given in theorem 2.

**Theorem 2.** If $x$ is the desired number of rows to be added to $P_n$ and $T_x$ is the total number of rows in $P_0$ that can be repositioned to make the addition of $x$ rows possible, then, $T_x = \lceil x(7.464) \rceil$ where $x = 1, 2, 3\ldots$

**Proof:**
Since the addition of a row is possible only every time 7.464 rows are reached then the addition of $x$ rows is possible only when 7.464 rows are reached $x$ times. Thus the ratio and proportion below hold for any $x$:

$$\frac{1}{7.464} = \frac{x}{T_x} \Rightarrow T_x = x(7.464)$$

$$= \lceil x(7.464) \rceil, \text{ since } T_x, x \in \mathbb{Z}^+$$

As illustrations, if one wishes to know how many rows are there in $P_0$ that can allow the insertion of 1, 2 and 3 more rows in $P_n$, then the desired totals are given below:

For $x = 1$: $T_1 = \lceil 7.464 \rceil = 14$ and 15

For $x = 2$: $T_2 = \lceil 2(7.464) \rceil = \lceil 14.928 \rceil = 15$

For $x = 3$: $T_2 = \lceil 3(7.464) \rceil = \lceil 21.393 \rceil = 23$

These illustrations show that the number of rows that can be inserted into a pack containing from 8 to 14 rows is the same and likewise the number of rows to be inserted into a pack containing from 15 to 22. In general, the number of rows to be inserted in $P_n$ is the same for the interval, $[\lceil x(7.464) \rceil]$, where $r$ is any number of rows in the interval.

**Number of Circles that can be Contained in $P_n$ after Repositioning**

The result above can be extended so that whenever the dimension of $P_0$ is known, then the maximum number of circles that can be fitted to $P_n$ can be determined.

The formula of computing the maximum number of circles to be fitted in $P_n$, based on the dimensions of $P_0$, is given in theorem 3.

**Theorem 3.** Let $r$ and $c$ be the number of rows and columns in $P_0$; $x$, the number of rows that can be inserted in $P_n$; $m$, the number of rows with maximum number of circles in $r$; and $k$, the number of rows with maximum number of circles in $x$. Then the total number ($T_n$) of circles that can be fitted in $P_n$ is

$$T_n = \frac{\sqrt{2}(r + x)(2c - 1)}{2}, \text{ if } r \text{ and } x \text{ are of the same parity}$$

$$\frac{\sqrt{2}(r + x)(2c - 1) + 1}{2}, \text{ if } r \text{ and } x \text{ are of different parity}$$

Where $x = \frac{r}{7.464}$

**Proof:**

Case 1. When $r$ and $x$ are of the same parity

If $r$ and $x$ are odd, the number ($m$) of rows with maximum number of circles in $r$ is greater by one than the number $(r - m)$ of rows with fewer circles; while in $x$, the number ($k$) of rows with maximum number of circles is one less than the number $(x - k)$ of rows with fewer circles. Thus, $m = (r - m) + 1 = \frac{1}{2}(r + 1)$ and $k = (x - k) - 1 = \frac{1}{2}(x - 1)$

In $r$, the number of circles in an odd row is $mc$ and in an even row is $(r - m)(c - 1)$. Similarly, in $x$, the number of circles in an odd row is $kc$ and in an even row is $(k - x)(c - 1)$. Let $T_n$ be the total number of circles in $P_n$. Then,

$$T_n = [mc + (r - m)(c - 1)] + [kc + (x - k)(c - 1)] = (mc + rc - mc + m) + (kc + xc - x - kc + k) = [r - r + (r + 1)/2] + [x - x + (x - )/2], \text{ since }$$

$$m = \frac{1}{2}(r + 1) \text{ and } k = \frac{1}{2}(x - 1)$$
The maximum number, $T_c$, of extra circles that can be inserted in $P_n$, based on the knowledge of the content of $P_o$, can be determined. The procedure of computing the value of $T_c$ is given in Theorem 4.

**Theorem 4:** If $r$ and $c$ are the rows and columns of $P_o$, $T_o$ is the number of circles in $P_o$ and $x$ is the number of rows that can be added to $P_n$, then the total number ($T$) of circles that can be inserted to $P_n$, given that $T_o = rc$, is

\[
T = T_o + T_c
\]

where $x =$ \( 7.464 \), as before.

**Corollary 1:** A positive value of $T_c$ gives the number of extra circles that can be inserted into $P_n$.

**Corollary 2:** A negative value of $T_c$ gives the number of rows that can contain the maximum number of circles.

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\[
T = T_o + T_c
\]

where $x =$ \( 7.464 \), as before.

**Corollary 1:** A positive value of $T_c$ gives the number of extra circles that can be inserted into $P_n$.
number of circles in $P_o$ that cannot be fitted in $P_n$.

**Corollary 3:** A zero value of $T_c$ means that the number of circles present in both $P_o$ and $P_n$ is the same.

**Minimum Number of Circles in a Row That Allows the Insertion of an Extra circle**

The addition of some extra circles is possible only if the number of rows in $P_n$ exceeds that of $P_o$. However, if the number of circles in the extra rows added is less than or equal to the number of circles displaced by the repositioning, then addition of some extra circles does not happen. Therefore it is important to determine the minimum number of circles in a row of $P_o$ that allows the addition of an extra circle whenever an extra row is added to $P_n$.

**Theorem 5.** If $r$ is the minimum number of rows in $P_o$ such that

$$\frac{r}{7.464} = 1$$

and $c$ is the minimum number of circles in the row of $P_o$ that allows the addition of one more circle to $P_n$, then $c = 5$.

**Proof:**

By hypothesis, the minimum number ($r$) of rows that satisfies the given equation is 8, a number that is shown previously to be also the minimum number of rows that allows the addition of one more row ($x = 1$) in $P_n$. Theorem 4 is used to compute the minimum number of circles that allows the addition of one more circle in $P_n$ by letting $T_c$ equal to 1. Thus,

$$T_c = \frac{1}{2} \cdot [x (2c - 1) + 1 - r],$$

since $r$ and $x$ are of different parity.

$$= \frac{1}{2} \cdot [1 (2c - 1) + 1 - 8],$$

since $x = 1$, $r = 8,$ and $T_c = 1$.

$$5 = c,$$

on solving

This means that when $P_o$ contains 8 rows and 5 columns one more circle can be added to $P_n$.

**Maximum Number of Circles That Can Be Fitted to a New Pack ($P_m$) by Partial Repositioning**

In the previous manner of repositioning the circles, addition of a row in $P_n$ is possible every time 7.464 rows are reached in $P_o$, as shown in theorems 1 and 2, and for these given rows addition of a circle in $P_n$ is possible every time a column is added in excess of four columns in $P_o$, as shown in theorems 4 and 5. For instance, one circle can be inserted to $P_n$ if $P_o$ contains 8 rows and 5 columns and for this given number of rows two circles can be inserted to $P_n$ if a column is added to $P_o$.

However, since the number of rows to be added to $P$ is the same for the interval $(7.464, 7.464 + 0.0000001)$ where $x$ is any integer, then repositioning the rows in excess of

$$\lfloor x(7.464) \rfloor$$

but less than $\lceil 7.464 \rceil$ result in the displacement of some circles because the alternate rows contain fewer circles.

Thus, in order to obtain the maximum number ($T_m$) of circles that can be fitted to a new pack ($P_m$) it is logical to reposition the number of rows equal to

$$\lceil x(7.464) \rceil$$

and to retain in their original positions the number of excess rows less than $\lceil 7.464 \rceil$. This situation is illustrated in figures 7(a), 7(b) and 7(c), where $P_o$ contains 50 circles with 10 rows and 5 columns.

The procedure of computing the maximum number ($T_m$) of circles that can be fitted to $P_m$ is given in theorem 6.

**Theorem 6.** Let $x$ be the number of rows that can be added to $P_n$, $r$ and $c$ be the number of rows and columns in $P_o$ and $T_x$ be a subset of $r$ that can allow the addition of $x$ rows. Then the number ($T_m$) of circles that can be fitted to a new pack ($P_m$) is maximum if the $T_x$ rows are the only ones repositioned. $T_m$ is given below:

$$T_m = \frac{1}{2} \cdot [2c(x+r)-(T_x+x)]$$

if $T_x$ and $x$ are of the same parity

$$= \frac{1}{2} \cdot [2c(x+r)-(T_x+x)+1]$$

if $T_x$ and $x$ are of different parity

Where, as before, $x = \frac{r}{7.464}$ and $T_x = \lceil x(7.464) \rceil$.

**Proof:**

By assumption, $T_x$ is the number of rows in $r$ that can allow the addition of $x$ more rows after repositioning and thus $r-T_x$ is the number of rows in $r$ that cannot allow the addition of a row by the
same action. The total number of circles in $T_x$ rows is given in Theorem 3, which is either 
$\frac{1}{2}(r + x)(2c-1)$ or $\frac{1}{2} [(r + x)(2c - 1) +1]$, depending on whether $x$ and $r$ are of the same or
different parity, and where $r$ is replaced by $T_x$. The
total number of circles in $r-T_x$ rows is $(r-T_x)c$
because this part is made to remain in its original
position. Thus, the total number ($T_m$) of circles that
can be fitted in $P_n$ by partial repositioning is

$$T_m = T_n + (r - T_x)c.$$  
Where:  
$T_n = \frac{1}{2}(T_x + x)(2c-1)$  or  
$= \frac{1}{2} [(T_x + x)(2c - 1) +1]$

Case 1. When $x$ and $T_x$ are of the same parity. 
$T_m = T_n + (r - T_x)c$
$= \frac{1}{2}(T_x + x)(2c-1) + (r - T_x)$  
c, by Theorem 3.  
$= \frac{1}{2} [2c(x+r) - (T_x + x)]$  on solving

Case 2. When $x$ and $T_x$ are of different parity 
$T_m = T_n + (r - T_x)c$
$= \frac{1}{2} [(T_x + x)(2c - 1) +1] + (r - T_x)$, by Theorem 3.  
$= \frac{1}{2} [2c(x+r) - (T_x + x) + 1]$  on solving

It now remains to show that $T_m \geq T_n$ for any 
column (c). It suffices to consider only one case, 
say, case 1.

Show that $T_m \geq T_n$. Assume that $r$ and $x$ are of 
the same parity. Then

$$T_m = \frac{1}{2}(r + x)(2c-1)$$
$= \frac{1}{2}[(T_x + x) + (r - T_x)](2c-1)$, since $r = T_x + (r - T_x)$

$$= \frac{1}{2}[(T_x + x)(2c-1) + (r - T_x)(2c-1)]$$
$\leq \frac{1}{2}[(T_x + x)(2c-1) + (r - T_x) 2c],$
since $(r - T_x) \geq 0$
$= \frac{1}{2}((2cT_x+2cx - T_x - x + 2cr - 2cT_x) = \frac{1}{2}(2c(x+r) - (T_x + x)]$
$= T_m$

Therefore $T_m$ is the maximum number of circles that can be fitted in $P_m$.

**Corollary 1:** $T_m > P_o$, if $x \geq 1$ and $c \geq 5$.

**Corollary 2:** $T_m = T_n$, if $r$ is a multiple of $T_x$.

**Corollary 3:** $T_m \geq T_n$, if $r$ is not a multiple of
$T_x$ and $x > 0$.

**Maximum Number of Circular objects that can
be Inserted in $P_m$**

The formula in determining the number ($T_c$) of 
circles that can be inserted in $P_m$ is derived similarly 
as the formula in determining the number of circles 
that can be inserted in $P_n$. Thus, if $x$ is the number of 
rows that can be added, $T_x$ is the number of rows 
that can allow the insertion of $x$ additional rows and 
r is the number of rows in $P_o$, then,

$$T_c = \frac{1}{2} [x(2c-1) - T_x],$$  if $x$ and $T_x$ are of the same parity 
$\frac{1}{2} [x(2c - 1) + (1 - T_x)],$  if $x$ and $T_x$ are of different 
parity

Where $x = \frac{r}{7.464}$, as before and

$$T_x = \left[ x(7.464) \right]$$

As illustrations, compute the maximum number 
of circular objects that can be fitted in the default
pack, $P_o$, new pack, $P_n$ and new pack, $P_m$, given that the rectangular container (or RC space) has dimensions, $r \times c = 20 \times 10$.

**Solutions**

1. The number of circular objects that can be fitted in $P_o$ is $T_o$.

   $T_o = rc = (20)(10) = 200$

2. The number of objects in $P_n$ is $T_n$.

   \[
   x = \frac{r}{7.464} = \frac{20}{7.464} = [2.680] = 2
   \]

   $T_n = \frac{1}{2}(r+x)(2c-1)$, since $r$ and $x$ are of the same parity
   \[
   = \frac{1}{2}(20+2)(2(10)-1)
   = \frac{1}{2}(22)(19)
   = 209
   \]

3. The number of objects in $P_m$ is $T_m$.

   $T_x = \left\lfloor x(7.464) \right\rfloor = \left\lfloor 2(7.464) \right\rfloor = \left\lfloor 14.928 \right\rfloor = 15$, since $x = 2$

   $T_m = \frac{1}{2} [ 2c (x + r) - (T_x + x) + 1 ]$, Since x and T_x are of different parity
   \[
   = \frac{1}{2} [ (2)(10) (2 +20) - (15 +2) + 1 ]
   = 212
   \]

4. $T_x = \frac{1}{2} [ x (2c - 1) + (1 - T_x) ]$, since $x$ and $T_x$ are of different parity: $x = 2$, $T_x = 15$.

   \[
   = \frac{1}{2} [ 2 (2(10) - 1) + (1- 15) ]
   = \frac{1}{2} [ 38 + (-14) ]
   = \frac{1}{2} (24)
   = 12
   \]

In the given examples, $P_n$ contains 9 objects more than that of $P_o$ and in turn, $P_m$ contains 3 objects more than that of $P_n$. Therefore, packing a rectangular container with circular objects by way of $P_m$ yields the maximum content of the container.

**Refinement of the Models**

A refinement of the above models of computing the maximum number of circular objects that can be contained in $P_n$, or the maximum number of objects that can be added to $P_n$, due to repositioning may be made by simply utilizing the fact about the minimum number of rows, $r$, in $P_o$ that allows the addition of one more row in $P_n$ and about the minimum number of columns, $c$, in $P_o$ that allows the addition of an object in $P_n$.

Since addition of a row in $P_n$ is possible every time the number of rows in $P_o$ is a multiple of 7.464 and addition of an object is possible whenever the number of columns, $c$, in $P_o$ is $c \geq 5$, then the total number, $T_a$, that can be added to $P_n$, and consequently the total number, $T_n$, of objects that can be contained in $P_n$ are the following:

\[
T_a = (c-4)( \frac{r}{7.464} )
\]

or if $x = \frac{r}{7.464}$

\[
T_a = (c-4)x \quad \text{and} \quad T_n = rc + T_a
\]

\[
T_n = c(r+x) - 4x
\]

From the model, it should be noted that $T_n$ contains maximum number if $r$ is equal to $\left\lfloor k(7.464) \right\rfloor$ for and $c \geq 5$. 

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CONCLUSIONS AND RECOMMENDATIONS

Based on from the results, this study shows that there are deterministic mathematical models of calculating the maximum number of identical circular objects that can be packed into rectangular spaces.

The models developed may be used to determine the maximum number of circular objects that can be packed in a given rectangular space, be that objects be vials in hospital, cigarette sticks in factories, pipes in hardware or planting materials in agriculture.

The mathematical models generated from the study may be used as bases in designing a rectangular container that can accommodate the maximum number of objects to be packed.

The mathematical procedures generated from the study may be a source of academic discourses in the academe, particularly serving as an example of a mathematical investigatory undertaking.

Since the study does not consider the situation where the rectangular space provides extra space along its dimensions in which case, necessary adjustment on the existing model has to be made, it is recommended that future study may consider such a case.

LITERATURE CITED


